V. *An Appendix to the Paper in the Philosophical Transactions for the Year* 1778, *Number* XLII, *pages* 902 *et seq. intitled,* "*A Method of extending* Cardan's "*Rule for resolving one Case of the Cubick Equation* "$x^3-qx=r$ *to the other Case of the same Equation,* "*which it is not naturally fitted to solve, and which is* "*therefore called* the irreducible Case." *By* Francis Maseres, *Esq. F. R. S. Cursitor Baron of the Exchequer.*

Read Nov. 4, 1779.

ARTICLE I.

IN the above-mentioned paper in the Philosophical Transactions the expression $\sqrt[3]{e}\times$ the infinite series $2+\frac{2ss}{9ee}-\frac{20s^4}{243e^4}+\frac{308s^6}{6561e^6}$ – &c. is shewn to be equal to the root of the equation $x^3-qx=r$, whenever $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, but greater than one half of it, or than $\frac{q^3}{54}$. This expression is wholly transcendental, or composed of an infinite number of terms, to wit, the terms of the series $2+\frac{2ss}{9ee}-\frac{20s^4}{243e^4}+\frac{308s^6}{6561e^6}$ – &c. multiplied into the cube-root of e. But I have since thought that it might be convenient on some occasions to divide this expression, if possible, into two others, whereof the one should be a mere algebraick expreffion, or confift of a finite number of terms, and the other fhould be tranfcendental, or involve in it an infinite feries. And I have accordingly difcovered a method of doing this, which I will now proceed to defcribe.

ART. 2. In the above-mentioned paper in the Philofophical Tranfactions I denoted the excefs of $\frac{q^3}{27}$ above $\frac{rr}{4}$ in the fecond cafe of the equation $x^3 - qx = r$, as well as the excefs of $\frac{rr}{4}$ above $\frac{q^3}{27}$ in the firft cafe of it, by the letters ss. But I have fince thought that it might have been better to denote the excefs of $\frac{q^3}{27}$ above $\frac{rr}{4}$ in the fecond cafe of that equation by the letters zz, in order the more clearly to diftinguifh it from the oppofite difference $\frac{rr}{4} - \frac{q^3}{27}$ in the firft cafe of it, which was denoted by ss. And I therefore in the courfe of the following pages fhall ufe the letters zz inftead of ss to denote the faid excefs of $\frac{q^3}{27}$ above $\frac{rr}{4}$ in the fecond cafe of the faid equation, or the difference $\frac{q^3}{27} - \frac{rr}{4}$.

ART. 3. Now, if zz be fubftituted inftead of ss in the expreffion $\sqrt{^3}e \times$ the infinite feries $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} -$ &c. that expreffion will thereby be converted into the following expreffion, to wit, $\sqrt{^3}e \times$ the infinite feries $2 + \frac{2zz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} -$ &c. Therefore, if $\frac{rr}{4}$ be lefs

lefs than $\frac{q^3}{27}$ but greater than $\frac{q^3}{54}$, and e be put $= \frac{r}{2}$, and zz be put $= \frac{q^3}{27} - \frac{rr}{4}$, the root of the equation $x^3 - qx = r$ will be equal to $\sqrt{}^3 e \times$ the infinite feries $2 + \frac{2zz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} -$ &c.

ART. 4. The numeral coefficients $\frac{2}{9}$, $\frac{20}{243}$, $\frac{308}{6561}$, &c. of $\frac{zz}{ee}$, $\frac{z^4}{e^4}$, $\frac{z^6}{e^6}$, &c. in this feries are exactly double of $\frac{1}{9}$, $\frac{10}{243}$, $\frac{154}{6561}$, &c. which are the numeral coefficients of the fame powers of the fraction $\frac{z}{e}$ in the feries $1 + \frac{z}{3e} - \frac{zz}{9ee} + \frac{5z^3}{81e^3} - \frac{10z^4}{243e^4} + \frac{22z^5}{729e^5} - \frac{154z^6}{6561e^6} + \frac{2618z^7}{137,781e^7} -$ &c. which is equal to the cube-root of the binomial quantity $1 + \frac{z}{e}$; or, if the numeral coefficients of the faid latter feries be denoted by the capital letters A, B, C, D, E, F, G, H, &c. refpectively, fo that A fhall be $= 1$, B $= \frac{1}{3}$, C $= \frac{1}{9}$, D $= \frac{5}{81}$, E $= \frac{10}{243}$, F $= \frac{22}{729}$, G $= \frac{154}{6561}$, and H $= \frac{2618}{137,781}$, and fo on, the faid numeral coefficients $\frac{2}{9}$, $\frac{20}{243}$, $\frac{308}{6561}$, &c. will be equal to 2C, 2E, 2G, &c. and the feries mentioned in the laft Article will be $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c. and confequently the root of the equation $x^3 - qx = r$, in the fecond cafe of it, in which $\frac{rr}{4}$ is lefs than $\frac{q^3}{27}$, will be equal to the expreffion $\sqrt{}^3 e \times$ the feries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c.

ART. 5. Now the feries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c. is is equal to the ſum of the two following ſerieſes, to wit, $2A - \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} - \frac{2Gz^6}{e^6} -$ &c. (in which all the terms following the firſt term are marked with the ſign $-$, or are ſubtracted from the firſt term), and $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} +$ &c; and the ſeries $2A - \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} - \frac{2Gz^6}{e^6} -$ &c. is equal to the ſum of the two ſerieſes $A + \frac{Bz}{e} - \frac{Czz}{ee} + \frac{Dz^3}{e^3} - \frac{Ez^4}{e^4} + \frac{Fz^5}{e^5} - \frac{Gz^6}{e^6} +$ &c. and $A - \frac{Bz}{e} - \frac{Czz}{ee} - \frac{Dz^3}{e^3} - \frac{Ez^4}{e^4} - \frac{Fz^5}{e^5} - \frac{Gz^6}{e^6} -$ &c. which are reſpectively equal to the cube-roots of the binomial quantities $1 + \frac{z}{e}$ and $1 - \frac{z}{e}$. Therefore the ſeries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c. is $= \sqrt[3]{1 + \frac{z}{e}} + \sqrt[3]{1 - \frac{z}{e}} +$ the infinite ſeries $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} +$ &c. Conſequently the expreſſion $\sqrt[3]{e} \times$ the ſeries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c. is $= \sqrt[3]{e} \times \sqrt[3]{1 + \frac{z}{e}} + \sqrt[3]{e} \times \sqrt[3]{1 - \frac{z}{e}} + \sqrt[3]{e} \times$ the infinite ſeries $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} +$ &c. $= \sqrt[3]{e+z} + \sqrt[3]{e-z} + \sqrt[3]{e} \times$ the infinite ſeries $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} +$ &c. $= \sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} +$ &c. Therefore the root of the equation $x^3 - qx = r$, in the ſecond caſe of it, in which $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, is equal to $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{14}}{e^{14}} +$ &c; of which expreſſion the firſt part, to

wit,

wit, $\sqrt[3]{e+z} + \sqrt[3]{e-z}$ is algebraick, and the latter part, to wit, $\sqrt[3]{e} \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} +$ &c. is tranſcendental. Q. E. I.[a]

Of the convergency of the Series obtained in the preceding Article.

ART. 6. This ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} +$ &c. evidently converges faſter than the ſeries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c. or $2 + \frac{2zz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} -$ &c: and conſequently the expreſſion $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} +$ &c. ſeems rather fitter

(a) N. B. I have been informed that both this mixed expreſſion of the root of the equation $x^3 - qx = r$ in the ſecond caſe of it, and the merely tranſcendental expreſſion of it publiſhed in the former paper, and from which this expreſſion is derived, were invented by Monſieur NICOLE, and publiſhed in the memoirs of the French Academy of Sciences ſo long ago as the year 1738; and the latter of them, to wit, the tranſcendental expreſſion $\sqrt[3]{e} \times$ the ſeries $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} -$ &c. I had myſelf ſeen many years ago in Monſieur CLAIRAUT's algebra, in the place cited in the 50th Article of my former paper, to wit, in pages 286, 287, 288. But it was obtained by the intervention of negative quantities, and the roots of negative quantities, which gave it, in my opinion, an air of great obſcurity. And therefore I thought an inveſtigation of the ſame ſeries, by a method that keeps clear of thoſe difficulties, might not be unacceptable to the lovers of theſe ſciences, nor unworthy of a place in the Tranſactions of this learned body.

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to exhibit the value of x in the equation $x^3 - qx = r$, in the second case of it (in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$), to a considerable degree of exactness than the former expression $\sqrt{}^3 e \times$ the series $2 + \frac{2zz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} -$ &c.

A Computation of the four first Terms of the Series obtained in Art. 5.

ART. 7. The first fifteen terms of the infinite series which is equal to the cube-root of $1 + \frac{z}{e}$ are as follows; to wit, $1 + \frac{z}{3e} - \frac{zz}{9ee} + \frac{5z^3}{81e^3} - \frac{10z^4}{243e^4} + \frac{22z^5}{729e^5} - \frac{154z^6}{6561e^6} + \frac{2618z^7}{137{,}781e^7} - \frac{935z^8}{59{,}049e^8} + \frac{21505z^9}{1{,}594{,}323e^9} - \frac{55913z^{10}}{4{,}782{,}969e^{10}} + \frac{147{,}407z^{11}}{14{,}348{,}907e^{11}} - \frac{1{,}179{,}256z^{12}}{129{,}140{,}163e^{12}} + \frac{3{,}174{,}920z^{13}}{387{,}420{,}489e^{13}} - \frac{60{,}323{,}480z^{14}}{8{,}135{,}830{,}269e^{14}} +$ &c; or, in decimal fractions, $1 +$ $.333{,}333{,}333$, &c. $\times \frac{z}{e} - .111{,}111{,}111$, &c. $\times \frac{zz}{ee} +$ $.061{,}728{,}395$, &c. $\times \frac{z^3}{e^3} - .041{,}152{,}263$, &c. $\times \frac{z^4}{e^4} +$ $.030{,}178{,}326$, &c. $\times \frac{z^5}{e^5} - .023{,}472{,}031$, &c. $\times \frac{z^6}{e^6} +$ $.019{,}001{,}167$, &c. $\times \frac{z^7}{e^7} - .015{,}834{,}305$, &c. $\times \frac{z^8}{e^8} +$ $.013{,}488{,}482$, &c. $\times \frac{z^9}{e^9} - .011{,}690{,}017$, &c. $\times \frac{z^{10}}{e^{10}} +$ $.010{,}273{,}045$, &c. $\times \frac{z^{11}}{e^{11}} - .009{,}131{,}595$, &c. $\times \frac{z^{12}}{e^{12}} +$ $.008{,}195{,}021$, &c. $\times \frac{z^{13}}{e^{13}} - .007{,}414{,}542$, &c. $\times \frac{z^{14}}{e^{14}}$. Therefore the four first terms of the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} +$

$\frac{Lz^{10}}{e^{10}}+\frac{Pz^{14}}{e^{14}}+\frac{Tz^{18}}{e^{18}}+$ &c. are $\frac{zz}{9ee}+\frac{154z^6}{6561e^6}+\frac{559132z^{10}}{4,782,969e^{10}}+\frac{60,323,480z^{14}}{8,135,830,269e^{14}}$, or, in decimal fractions, .111,111,111, &c. $\times \frac{zz}{ee}+$ .023,472,031, &c. $\times \frac{z^6}{e^6}+$ .011,690,017, &c. $\times \frac{z^{10}}{e^{10}}+$ .007,414,542, &c. $\times \frac{z^{14}}{e^{14}}$. Therefore the root of the cubick equation $x^3-qx=r$, in the second case of it (in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$), is equal to $\sqrt[3]{e+z}+\sqrt[3]{e-z}+4\sqrt[3]{e}\times$ the series $\frac{zz}{9ee}+\frac{154z^6}{6561e^6}+\frac{55,913z^{10}}{4,782,969e^{10}}+\frac{60,323,480z^{14}}{8,135,830,269e^{14}}+$ &c. *ad infinitum*, or $\sqrt[3]{e+z}+\sqrt[3]{e-z}+4\sqrt[3]{e}\times$ the series .111,111,111, &c. $\times \frac{zz}{ee}+$ .023,472,031, &c. $\times \frac{z^6}{e^6}+$ .011,690,017, &c. $\times \frac{z^{10}}{e^{10}}+$ .007,414,542, &c. $\times \frac{z^{14}}{e^{14}}+$ &c. *ad infinitum.*

Of the best Manner of Proceeding to the Computation of more Terms of the said Series, if required.

ART. 8. If more than four terms of this last series are required, it will be necessary to compute the series $1+\frac{z}{3e}-\frac{zz}{9ee}+\frac{5z^3}{81e^3}-\frac{10z^4}{243e^4}+\frac{22z^5}{729e^5}-\frac{154z^6}{6561e^6}+$ &c. (which is $=\sqrt[3]{1+\frac{z}{e}}$), to more than fifteen terms; in order to which it will be convenient to express the terms of that series in the following manner, to wit, $1+\frac{1}{3}\times\frac{Az}{e}-\frac{2}{6}\times\frac{Bzz}{ee}+\frac{5}{9}\times\frac{Cz^3}{e^3}-\frac{8}{12}\times\frac{Dz^4}{e^4}+\frac{11}{15}\times\frac{Ez^5}{e^5}-\frac{14}{18}\times\frac{Fz^6}{e^6}+\frac{17}{21}\times\frac{Gz^7}{e^7}-\frac{20}{24}\times\frac{Hz^8}{e^8}+\frac{23}{27}\times\frac{Iz^9}{e^9}-\frac{26}{30}\times\frac{Kz^{10}}{e^{10}}+\frac{29}{33}\times$

N 2 $\frac{Lz^{11}}{e^{11}}$

$\frac{Lz^{11}}{e^{11}} - \frac{32}{36} \times \frac{Mz^{12}}{e^{12}} + \frac{35}{39} \times \frac{Nz^{13}}{e^{13}} - \frac{38}{42} \times \frac{Oz^{14}}{e^{14}} + \frac{41}{45} \times \frac{Pz^{15}}{e^{15}} - \frac{44}{48} \times \frac{Qz^{16}}{e^{16}} + \frac{47}{51} \times \frac{Rz^{17}}{e^{17}} - \frac{50}{54} \times \frac{Sz^{18}}{e^{18}} + \frac{53}{57} \times \frac{Tz^{19}}{e^{19}} - \frac{56}{60} \times \frac{Vz^{20}}{e^{20}} + \frac{59}{63} \times \frac{Wz^{21}}{e^{21}} - \frac{62}{66} \times \frac{Xz^{22}}{e^{22}}$ + &c. or $A + \frac{Bz}{e} - \frac{Czz}{ee} + \frac{Dz^3}{e^3} - \frac{Ez^4}{e^4} + \frac{Fz^5}{e^5} - \frac{Gz^6}{e^6} + \frac{Hz^7}{e^7} - \frac{Iz^8}{e^8} + \frac{Kz^9}{e^9} - \frac{Lz^{10}}{e^{10}} + \frac{Mz^{11}}{e^{11}} - \frac{N^{12}}{e^{12}} + \frac{Oz^{13}}{e^{13}} - \frac{Pz^{14}}{e^{14}} + \frac{41}{45} \times \frac{Pz^{15}}{e^{15}} - \frac{44}{48} \times \frac{Qz^{16}}{e^{16}} + \frac{47}{51} \times \frac{Rz^{17}}{e^{17}} - \frac{50}{54} \times \frac{Sz^{18}}{e^{18}} + \frac{53}{57} \times \frac{Tz^{19}}{e^{19}} - \frac{56}{60} \times \frac{Vz^{20}}{e^{20}} + \frac{59}{63} \times \frac{Wz^{21}}{e^{21}} - \frac{62}{66} \times \frac{Xz^{22}}{e^{22}}$ + &c; in which it is evident that the generating fractions of the coefficients of the several terms are derived from thofe that immediately precede them by the continual addition of the number 3 to both their numerators and denominators.

An example of the refolution of a cubick equation by means of the expreffion $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ *the feries* $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}}$ + *&c. given in Art.* 5.

ART. 9. Let it be required to refolve the equation $x^3 - x = \frac{1}{3}$ by means of the faid expreffion.

Here q is $= 1$, and $r = \frac{1}{3}$; and confequently q^3 is $= 1$, and $\frac{q^3}{27} = \frac{1}{27}$, and $\frac{r}{2} = \frac{1}{6}$, and $\frac{rr}{4} = \frac{1}{36}$, which is lefs than $\frac{1}{27}$, or $\frac{q^3}{27}$. Therefore this equation does not come under CARDAN's rule, but may be refolved by the expreffion given in Art. 5, provided that $\frac{rr}{4}$, though lefs than $\frac{q^3}{27}$, is greater than half $\frac{q^3}{27}$, or than $\frac{q^3}{54}$; which it is, becaufe it is $=\frac{1}{36}$, whereas $\frac{q^3}{54}$ is equal only to $\frac{1}{54}$, which is leſs than $\frac{1}{36}$. Therefore the propoſed equation $x^3-x=\frac{1}{3}$ may be reſolved by means of the ſaid expreſſion.

ART. 10. Now, ſince in this caſe q is $=1$, and r is $=\frac{1}{3}$, we ſhall have $\frac{r}{2}$, or e, $=\frac{1}{6}$, and $\frac{q^3}{27}-\frac{rr}{4}$ $(=\frac{1}{27}-\frac{1}{36}=\frac{4}{108}-\frac{3}{108}=\frac{1}{108})=\frac{1}{36\times 3}$; that is, zz will be $=\frac{1}{36\times 3}$, and conſequently z will be $=\frac{1}{6\sqrt{3}}$. Therefore $e+z$ will be $=\frac{1}{6}+\frac{1}{6\sqrt{3}}$ $(=\frac{\sqrt{3}}{6\sqrt{3}}+\frac{1}{6\sqrt{3}}=\frac{3}{6\times 3}+\frac{\sqrt{3}}{6\times 3}=\frac{3+\sqrt{3}}{6\times 3}=\frac{3+\sqrt{3}}{18}=\frac{3+1.732,050,8}{18}=\frac{4.732,050,8}{18})=.262,891,71$; and $e-z$ will be $=\frac{1}{6}-\frac{1}{6\sqrt{3}}$ $(=\frac{3-\sqrt{3}}{18}=\frac{3-1.732,050,8}{18}=\frac{1.267,949,2}{18})=.070,441,62$. Therefore the cube-root of $e+z$ is $=\sqrt[3]{.262,891,71}=.640,607,91$; and the cube-root of $e-z$ is $=\sqrt[3]{.070,441,62}=.412,993,40$; and conſequently $\sqrt[3]{e+z}+\sqrt[3]{e-z}$ is $=.640,607,91+.412,993,40=1.053,601,31$.

ART. 11. It remains that we compute the infinite ſeries $\frac{Czz}{ee}+\frac{Gz^6}{e^6}+\frac{Lz^{10}}{e^{10}}+\frac{Pz^{14}}{e^{14}}+\frac{Tz^{16}}{e^{16}}$ + &c. and extract the cube-root of e, and then multiply the ſaid ſeries into 4 times the ſaid cube-root.

Now the cube-root of e is in this caſe $=\sqrt[3]{\frac{1}{6}}=\frac{1}{\sqrt[3]{6}}=\frac{1}{1.817,121}$; and conſequently $4\sqrt[3]{e}$ is $=\frac{4}{1.817,121}$.

And,

And, since zz is $= \frac{1}{36 \times 3}$, and ee is $= \frac{1}{36}$, it follows that $\frac{zz}{ee}$ will be $= \frac{1}{3}$. Therefore $\frac{z^4}{e^4}$ will be $= \frac{1}{9}$, and $\frac{z^6}{e^6} = \frac{1}{27}$, and $\frac{z^{10}}{e^{10}}$ $(= \frac{z^6}{e^6} \times \frac{z^4}{e^4} = \frac{1}{27} \times \frac{1}{9}) = \frac{1}{243}$, and $\frac{z^{14}}{e^{14}}$ $(= \frac{z^{10}}{e^{10}} \times \frac{z^4}{e^4} = \frac{1}{243} \times \frac{1}{9}) = \frac{1}{2187}$. Therefore the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}}$ + &c. will in this case be $= \frac{C}{3} + \frac{G}{27} + \frac{L}{243} + \frac{P}{2187}$ + &c. $= \frac{.111,111,111, \&c.}{3} + \frac{.023,472,031, \&c.}{27} + \frac{.011,690,017, \&c.}{243} + \frac{.007,414,542, \&c.}{2187}$ + &c. = .037,037,037,&c. + .000,869,334,&c. + 000,048,107, &c. + 000,003,390, &c. + &c. = .037,957,868, &c.

Therefore $4\sqrt{^3}e \times$ the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}}$ + &c. is equal to $\frac{4}{1.817,121} \times .037,957,868$, &c. $= \frac{.151,831,472, \&c.}{1.817,121} =$ .083,556,06.

Consequently the whole expression $\sqrt{^3}\overline{e+z} + \sqrt{^3}\overline{e-z} + 4\sqrt{^3}e \times$ the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}}$ + &c. is = 1.053,601,31 + .083,556,06 = 1.137,157,37; that is, the root of the proposed equation $x^3 - x = \frac{1}{3}$ is = 1.137,157,37. Q. E. I.

ART. 12. This value of the root of the equation $x^3 - x = \frac{1}{3}$ is exact to six places of figures, the more accurate value of it being 1.137,158,164. We may therefore conclude that the expression here made use of to determine the value of x, to wit, $\sqrt{^3}\overline{e+z} + \sqrt{^3}\overline{e-z} +$

$4\sqrt{^3}e \times$

$4\sqrt{^3e}$ × the infinite ſeries $\frac{Czz}{ee}+\frac{Gz^6}{e^6}+\frac{Lz^{10}}{e^{10}}+\frac{Pz^{14}}{e^{14}}+$ &c. is ſomewhat preferable, with reſpect to the practical reſolution of theſe equations, to the other expreſſion of its value given in the former paper in the Philoſophical Tranſactions, to wit, $\sqrt{^3e}$ × the infinite ſeries $2+\frac{2zz}{9ee}-\frac{20z^4}{243e^4}+\frac{308z^6}{6561e^6}-$ &c. For it appeared in Art. 42 of that paper, page 941, that the value of the root of this ſame equation $x^3-x=\frac{1}{3}$ obtained by computing four terms of the ſeries $2+\frac{2zz}{9ee}-\frac{20z^4}{243e^4}+\frac{308z^6}{6561e^6}-$ &c. was 1.137,33; which is true only to four places of figures; whereas by computing the ſame number of terms of the ſeries $\frac{Czz}{ee}+\frac{Gz^6}{e^6}+\frac{Lz^{10}}{e^{10}}+\frac{Pz^{14}}{e^{14}}+$ &c. we have juſt now obtained a value of the ſame root, to wit, the number 1.137,157,37, which is exact to ſix places of figures. This is agreeable to what was obſerved above in Art. 6.

A Summary of the Concluſions obtained in this Paper and the former Paper to which it is an Appendix.

ART. 13. I will now conclude this paper by ſetting down, in as conciſe a manner as I can, the ſeveral concluſions that have been obtained in this and the abovementioned paper in the Philoſophical Tranſactions for the

the year 1778, Number XLII. page 902, &c. concerning the root of the cubick equation $x^3-qx=r$, that the whole may be ſeen together at one view.

ART. 14. If $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and e be put $=\frac{r}{2}$, and $ss=\frac{rr}{4}-\frac{q^3}{27}$, it is ſhewn in Art. 5, of the ſaid former paper, that the root of the equation $x^3-qx=r$ will be $=\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}\;\frac{+q}{3\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}}$, or $\sqrt[3]{e+s}+\frac{q}{3\sqrt[3]{e+s}}$.

ART. 15. And it is ſhewn in Art. 9. of the ſaid paper, that the root of the ſaid equation will in that caſe be alſo equal to $\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}\;\frac{q}{3\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}}$, or $\sqrt[3]{e-s}+\frac{q}{3\sqrt[3]{e-s}}$.

ART. 16. And it is ſhewn in Art. 11, of the ſaid paper, page 915, that the root of the ſaid equation will in that caſe be alſo equal to $\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}+\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}$, or $\sqrt[3]{e+s}+\sqrt[3]{e-s}$.

ART. 17. And it is ſhewn in Art. 23, of the ſaid paper, page 923, that the root of the ſaid equation will in that caſe be alſo equal to $\sqrt[3]{e}\times$ the infinite ſeries $2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-\frac{1870s^8}{59049e^8}-\frac{111,826s^{10}}{4,782,969e^{10}}-\frac{2,358,512s^{12}}{129,140,163e^{12}}-\frac{120,646,960s^{14}}{8,135,830,269e^{14}}$

$\frac{120,646,960s^{14}}{8,135,830,269e^{14}}$ – &c. or (if we put the capital letters A, B, C, D, E, F, G, H, &c. for the several numeral coefficients 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{5}{81}$, $\frac{10}{243}$, $\frac{22}{729}$, $\frac{154}{6561}$, $\frac{2618}{137,781}$, &c. of the terms of the series $1+\frac{s}{3e}-\frac{ss}{9ee}+\frac{5s^3}{81e^3}-\frac{10s^4}{243e^4}+\frac{22s^5}{729e^5}-\frac{154s^6}{6561e^6}+\frac{2618s^7}{137,781e^7}-$ &c. which is equal to the cube-root of the binomial quantity $1+\frac{s}{e}$), equal to $\sqrt{^3}e\times$ the infinite series $2A-\frac{2Css}{ee}-\frac{2Es^4}{e^4}-\frac{2Gs^6}{e^6}-\frac{2Is^8}{e^8}-\frac{2Ls^{10}}{e^{10}}-\frac{2N^{12}}{e^{12}}-\frac{2P^{14}}{e^{14}}-$ &c; in which series all the terms after the first term 2A, or 2, are marked with the sign –, or are to be subtracted from the said first term.

ART. 18. And, if $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, but greater than its half, or $\frac{q^3}{54}$, and e be put, as before, $=\frac{r}{2}$, and $zz=\frac{q^3}{27}-\frac{rr}{4}$, it is shewn in Art. 31, 32, 33, 34, 35, of the said paper, pages 927—936, that the root of the equation $x^3-qx=r$ will be equal to $\sqrt{^3}e\times$ the infinite series $2+\frac{2zz}{9ee}-\frac{20z^4}{243e^4}+\frac{308z^6}{6561e^6}-\frac{1870z^8}{59049e^8}+\frac{111,826z^{10}}{4,782,969e^{10}}-\frac{2,358,512z^{12}}{129,140,163e^{12}}+\frac{120,646,960z^{14}}{8,135,830,269e^{14}}-$ &c. or $\sqrt{^3}e\times$ the infinite series $2A+\frac{2Czz}{ee}-\frac{2Ez^4}{e^4}+\frac{2Gz^6}{e^6}-\frac{2Iz^8}{e^8}+\frac{2Lz^{10}}{e^{10}}-\frac{2Nz^{12}}{e^{12}}+\frac{2Pz^{14}}{e^{14}}-\frac{2Rz^{16}}{e^{16}}+\frac{2Tz^{18}}{e^{18}}-$&c; in which series the capital letters A, C, E, G, I, L, N, P, R, T, &c. denote the same numeral coefficients 1, $\frac{1}{9}$, $\frac{10}{243}$, $\frac{154}{6561}$, &c. as in the last Article, and all the terms that involve the odd powers of the fraction $\frac{zz}{ee}$, to wit, $\frac{zz}{ee}$ itſelf, $\overline{\frac{zz}{ee}}|^3$, or $\frac{z^6}{e^6}$, $\overline{\frac{zz}{ee}}|^5$, or $\frac{z^{10}}{e^{10}}$, $\overline{\frac{zz}{ee}}|^7$, or $\frac{z^{14}}{e^{14}}$, $\overline{\frac{zz}{ee}}|^9$, or $\frac{z^{18}}{e^{18}}$, &c. are marked with the ſign +, or are to be added to the firſt term 2A, or 2, and all the other terms are marked with the ſign −, or are to be ſubtracted from the former.

Theſe are the concluſions obtained in the ſaid former paper, which is printed in the Philoſophical Tranſactions for the year 1778, Number XLII. pages 902—949. The concluſions obtained in this paper are as follows.

ART. 19. If $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, but greater than its half, or $\frac{q^3}{54}$, and e be put, as before, $= \frac{r}{2}$, and $zz = \frac{q^3}{27} - \frac{rr}{4}$, it is ſhewn in the preſent paper, Art. 5 and 7, that the root of the equation $x^3 - qx = r$ will be equal to the mixed expreſſion $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the infinite ſeries $\frac{zz}{9ee} + \frac{154z^6}{6561e^6} + \frac{55913z^{10}}{4,782,969e^{10}} + \frac{60,323,480z^{14}}{8,135,830,269e^{14}}$ + &c. or $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the infinite ſeries .111,111,111, &c. $\times \frac{zz}{ee}$ + .023,472,031, &c. $\times \frac{z^6}{e^6}$ + .011,690,017, &c. $\times \frac{z^{10}}{e^{10}}$ + .007,414,542, &c. $\times \frac{z^{14}}{e^{14}}$ + &c. or $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the infinite ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}}$ + &c; in which ſeries the capital letters C, G, L, P, &c. denote the same numeral coefficients $\frac{1}{9}$, $\frac{154}{6561}$, $\frac{55913}{4,782,969}$, $\frac{60,323,480}{8,135,830,269}$, &c. as they denoted in the two last Articles, to wit, the coefficients of $\frac{zz}{ee}$, $\frac{z^6}{e^6}$, $\frac{z^{10}}{e^{10}}$, $\frac{z^{14}}{e^{14}}$, $\frac{z^{18}}{e^{18}}$, &c. or of the odd powers of $\frac{zz}{ee}$ in the series which is equal to $\sqrt[3]{1+\frac{z}{e}}$. And it is also shewn in Art. 6 and 12 of this paper, that this expression, computed to a given number of terms, gives the value of x somewhat more exactly than the former expression, $\sqrt[3]{e} \times$ the infinite series $2+\frac{2zz}{9ee}-\frac{20z^4}{243e^4}+\frac{308z^6}{6561e^6}-$ &c. if computed only to the same number of terms.

ART. 20. As to the second branch of the second case of the equation $x^3-qx=r$, or that in which $\frac{rr}{4}$ is less than half $\frac{q^3}{27}$, or than $\frac{q^3}{54}$, I do not know any method of extending CARDAN's rule to it. But I have been informed by my learned and ingenious friend Dr. CHARLES HUTTON, Professor of Mathematicks in the Royal Academy at Woolwich, that he has discovered such a method: and I hope he will soon communicate it to this learned Society.